Title: Application of symmetries to differential equations: symmetry reduction and solution transformation examples

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# Application of symmetries to differential equations: symmetry reduction and solution transformation examples

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In the following lecture notes, we provide an outline of what we mean by symmetries of differential equations and illustrate their utility in simplification of differential equations which may lead to solutions. Loosely, a symmetry of a differential equation is a local group of transformations acting on the space of independent and dependent variables that transform solutions of a differential equation to new solutions, i.e., if

$$\Delta(\mathbf{x}, \mathbf{u}, \mathbf{p}) = 0, \tag{1}$$

denotes a differential equation with a solution prescribed by $[\mathbf{x}, \mathbf{u}, \mathbf{p}]$, then knowledge of a symmetry facilitates the attainment of a new solution, $[\tilde{\mathbf{x}}, \tilde{\mathbf{u}}, \tilde{\mathbf{p}}]$, such that

$$\Delta(\mathbf{x}, \mathbf{u}, \mathbf{p}) = \Delta(\tilde{\mathbf{x}}, \tilde{\mathbf{u}}, \tilde{\mathbf{p}}) = 0. \tag{2}$$

In Equations (1) and (2), $\mathbf{x}$, $\mathbf{u}$ and $\mathbf{p}$ denote the independent variables, dependent variables and partial derivatives, respectively and the tildes denote the transformations. Equation (2) will subsequently be referred to as the symmetry criterion.

We can apply the knowledge of the existence of a symmetry in two ways,

1. If no solution is known apriori, the symmetry can be applied to simplify the differential equation through the method of symmetry reduction. Often, the resultant simplification may enable a solution to be determined.

2. If we already possess a solution to a differential equation, knowledge of an admissible symmetry enables us to generate new solutions.

## 1 Symmetry reduction of differential equations

### 1.1 Example 1: The heat equation

The one-dimensional heat equation is given by

$$\Delta(\mathbf{x}, \mathbf{u}, \mathbf{p}) = \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \tag{3}$$

where $x$, $t$ and $u$ denote space, time and velocity, respectively and the thermal diffusivity has been normalized to one. For this example,

$$\mathbf{x} = [x, t], \qquad \mathbf{u} = u, \qquad \mathbf{p} = \left[\frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial x^2}\right]. \tag{4}$$

Equation (3) possesses the scaling/translation symmetry

$$\tilde{x} = x, \qquad \tilde{t} = t + \epsilon, \qquad \tilde{u} = ue^{a\epsilon}, \tag{5}$$

where $a$ is a constant, $\epsilon$ is the group parameter and the tildes denote transformed variables.

**Remark 1.** *Note that the identity transformation corresponds to the group parameter, $\epsilon$, being equal to zero, i.e.*

$$\tilde{x}\big|_{\epsilon=0} = x, \qquad \tilde{t}\big|_{\epsilon=0} = t, \qquad \tilde{u}\big|_{\epsilon=0} = u. \tag{6}$$

We can show that Equation (5) is a symmetry of the heat equation by substituting in the transformation for $u$, expanding via the chain rule and checking that Equation (2) is satisfied. Substituting for $u$ in Equation (3) yields

$$\frac{\partial(e^{-a\epsilon}\tilde{u})}{\partial t} - \frac{\partial^2(e^{-a\epsilon}\tilde{u})}{\partial x^2} = 0. \tag{7}$$

Treating $\tilde{u}$ as a function of $\tilde{t}$ and $\tilde{x}$, this equation is expanded to

$$\frac{\partial\tilde{t}}{\partial t}\frac{\partial(e^{-a\epsilon}\tilde{u})}{\partial\tilde{t}} + \frac{\partial\tilde{x}}{\partial x}\frac{\partial(e^{-a\epsilon}\tilde{u})}{\partial\tilde{x}} - \frac{\partial}{\partial x}\left(\frac{\partial\tilde{t}}{\partial x}\frac{\partial(e^{-a\epsilon}\tilde{u})}{\partial\tilde{t}} + \frac{\partial\tilde{x}}{\partial x}\frac{\partial(e^{-a\epsilon}\tilde{u})}{\partial\tilde{x}}\right) = 0. \tag{8}$$

From the transformations in Equation (5), the following first order derivatives are computed

$$\frac{\partial\tilde{t}}{\partial t} = 1, \qquad \frac{\partial\tilde{t}}{\partial x} = 0, \qquad \frac{\partial\tilde{x}}{\partial t} = 0, \qquad \frac{\partial\tilde{x}}{\partial x} = 1. \tag{9}$$

Thus, Equation (8) simplifies to

$$e^{-a\epsilon}\frac{\partial\tilde{u}}{\partial\tilde{t}} - \frac{\partial}{\partial x}\left(e^{-a\epsilon}\frac{\partial\tilde{u}}{\partial\tilde{x}}\right) = 0. \tag{10}$$

Further expansion of the second term then gives

$$e^{-a\epsilon}\left(\frac{\partial\tilde{u}}{\partial\tilde{t}} - \frac{\partial\tilde{t}}{\partial x}\frac{\partial^2\tilde{u}}{\partial\tilde{t}\partial\tilde{x}} - \frac{\partial\tilde{x}}{\partial x}\frac{\partial^2\tilde{u}}{\partial\tilde{x}^2}\right) = 0. \tag{11}$$

which simplifies to

$$\frac{\partial\tilde{u}}{\partial\tilde{t}} - \frac{\partial^2\tilde{u}}{\partial\tilde{x}^2} = \Delta(\tilde{\mathbf{x}}, \tilde{\mathbf{u}}, \tilde{\mathbf{p}}) = 0. \tag{12}$$

Thus, Equation (2) is satisfied and Equation (5) represents a symmetry of the heat equation.

### 1.1.1 Symmetry reduction of the heat equation

Our first step towards reducing Equation (3) using the known symmetry involves expanding the transformations of the variables as a Taylor series around the identity transformation, $\epsilon = 0$

$$\begin{aligned}
\tilde{x} &= \tilde{x}\big|_{\epsilon=0} + \epsilon\left.\frac{\partial\tilde{x}}{\partial\epsilon}\right|_{\epsilon=0} + \mathcal{O}(\epsilon^2), && (13)\\
&= x + \epsilon\left.\frac{\partial\tilde{x}}{\partial\epsilon}\right|_{\epsilon=0} + \mathcal{O}(\epsilon^2), && (14)\\
&= x + \mathcal{O}(\epsilon^2), && (15)\\
\tilde{t} &= t + \epsilon\left.\frac{\partial\tilde{t}}{\partial\epsilon}\right|_{\epsilon=0} + \mathcal{O}(\epsilon^2), && (16)\\
&= t + \epsilon + \mathcal{O}(\epsilon^2), && (17)\\
\tilde{u} &= u + \epsilon\left.\frac{\partial\tilde{u}}{\partial\epsilon}\right|_{\epsilon=0} + \mathcal{O}(\epsilon^2), && (18)\\
&= u + \epsilon a u + \mathcal{O}(\epsilon^2). && (19)
\end{aligned}$$

Subtracting the original variables from the transformed variables and dividing by $\epsilon$ yields

$$\frac{\tilde{x} - x}{\epsilon} = \mathcal{O}(\epsilon), \qquad \frac{\tilde{t} - t}{\epsilon} = 1 + \mathcal{O}(\epsilon), \qquad \frac{\tilde{u} - u}{\epsilon} = au + \mathcal{O}(\epsilon). \tag{20}$$

In the limit $\epsilon \to 0$, these become

$$\frac{\mathrm{d}x}{\mathrm{d}\epsilon} = 0, \qquad \frac{\mathrm{d}t}{\mathrm{d}\epsilon} = 1, \qquad \frac{\mathrm{d}u}{\mathrm{d}\epsilon} = au, \tag{21}$$

assuming each Taylor series converges. Solving each for $\mathrm{d}\epsilon$, results in the characteristic system

$$\frac{\mathrm{d}t}{1} = \frac{\mathrm{d}x}{0} = \frac{\mathrm{d}u}{au} = \mathrm{d}\epsilon. \tag{22}$$

The characteristic system can be solved to determine a set of new variables. Solving the equation consisting of the first and second members

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 0, \tag{23}$$

yields

$$r = x, \tag{24}$$

where $r$ is the constant of integration and defines the first new coordinate. Then, from the equation comprised of the first and third members

$$\int a\mathrm{d}t = \int \frac{\mathrm{d}u}{u}, \tag{25}$$

we obtain

$$w = ue^{-at}, \tag{26}$$

where $w$ is the constant of integration and is the second new coordinate. Finally, from the first and last members of the characteristic system we have

$$\int \mathrm{d}t = \int \mathrm{d}\epsilon, \tag{27}$$

and therefore

$$t = \epsilon + c, \tag{28}$$

from which we define the shift coordinate

$$s = t. \tag{29}$$

Summarizing

$$r = x, \qquad s = t, \qquad w = ue^{-at}. \tag{30}$$

The variables $r$, $s$ and $w$ constitute a new set of variables defining a new coordinate system which we will refer to as $rsw$-space. We can see how the transformations given by Equation (5), originally prescribed in $xtu$-space, transfer to $rsw$-space

$$\tilde{r} = \tilde{x} = x = r, \qquad \text{(Identity)}, \tag{31}$$

$$\tilde{s} = \tilde{t} = t + \epsilon = s + \epsilon, \qquad \text{(Translation)}, \tag{32}$$

$$\tilde{w} = \tilde{u}e^{-a\tilde{t}} = ue^{a\epsilon}e^{-at-a\epsilon} = ue^{-at} = w, \qquad \text{(Identity)}. \tag{33}$$

Thus, in $rsw$-space the characteristic system is

$$\frac{\mathrm{d}r}{0} = \frac{\mathrm{d}s}{1} = \frac{\mathrm{d}w}{0} = \mathrm{d}\epsilon. \tag{34}$$

Now we wish to re-express the heat equation in terms of the new $rsw$ variables. We begin by substituting for $u$ to obtain

$$\frac{\partial we^{at}}{\partial t} - \frac{\partial^2(we^{at})}{\partial x^2} = 0. \tag{35}$$

Expanding using the chain rule gives

$$w\frac{\partial e^{at}}{\partial t} + e^{at}\frac{\partial w}{\partial t} - e^{at}\frac{\partial^2 w}{\partial x^2} = 0, \tag{36}$$

which simplifies to

$$e^{at}\left(aw + \frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2}\right) = 0, \tag{37}$$

and therefore

$$aw + \frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} = 0. \tag{38}$$

Treating $w$ as a function of $r$ and $s$, the chain rule then yields

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial w}{\partial s}\frac{\partial s}{\partial t}, \tag{39}$$

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial w}{\partial s}\frac{\partial s}{\partial x}. \tag{40}$$

From the relationships given in Equation (30),

$$\frac{\partial r}{\partial t} = 0, \qquad \frac{\partial r}{\partial x} = 1, \qquad \frac{\partial s}{\partial t} = 1, \qquad \frac{\partial s}{\partial x} = 0, \tag{41}$$

and therefore the first derivatives of $w$ simplify to

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial s}, \tag{42}$$

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}. \tag{43}$$

The second spatial derivative is then simply

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial r}\right), \tag{44}$$

$$= \frac{\partial r}{\partial x}\frac{\partial^2 w}{\partial r^2} + \frac{\partial s}{\partial x}\frac{\partial^2 w}{\partial s \partial r}, \tag{45}$$

$$= \frac{\partial^2 w}{\partial r^2}. \tag{46}$$

The heat equation, expressed in $rsw$-space is then

$$aw + \frac{\partial w}{\partial s} - \frac{\partial^2 w}{\partial r^2} = 0. \tag{47}$$

We refer to this equation as the target equation.

Now that we have determined the target equation, we proceed to simplify it using knowledge of the $rsw$-characteristic system given in Equation (34). Along the direction $\mathrm{d}\epsilon$, the rate of change of $w$ is given by

$$\frac{\mathrm{d}w}{\mathrm{d}\epsilon} = \frac{\partial w}{\partial r}\frac{\mathrm{d}r}{\mathrm{d}\epsilon} + \frac{\partial w}{\partial s}\frac{\mathrm{d}s}{\mathrm{d}\epsilon}. \tag{48}$$

From the characteristic system, we have

$$\frac{\mathrm{d}w}{\mathrm{d}\epsilon} = 0, \qquad \frac{\mathrm{d}r}{\mathrm{d}\epsilon} = 0, \qquad \frac{\mathrm{d}s}{\mathrm{d}\epsilon} = 1. \tag{49}$$

Substituting these relationships into Equation (48) yields

$$\frac{\partial w}{\partial s} = 0. \tag{50}$$

Thus, the target equation is simplified to the reduced equation

$$aw - \frac{\partial^2 w}{\partial r^2} = 0, \tag{51}$$

along the direction d$\epsilon$.

Equation (51) is a second-order linear ordinary differential equation and can be solved by assuming $w$ is proportional to $e^{\lambda r}$ where $\lambda$ is some constant. Thus

$$w \propto e^{\lambda r}, \tag{52}$$

and

$$\frac{\partial^2 w}{\partial r^2} \propto \lambda^2 e^{\lambda r}. \tag{53}$$

Substituting into Equation (51) we find

$$a e^{\lambda r} - \lambda^2 e^{\lambda r} = 0 = e^{\lambda r}\left(a - \lambda^2\right), \tag{54}$$

therefore we require

$$a = \lambda^2, \quad \text{or equivalently} \quad \lambda = \pm\sqrt{a}. \tag{55}$$

The solutions to Equation (51) are therefore

$$w(r) = Ae^{-\sqrt{a}r}, \qquad w(r) = Be^{\sqrt{a}r}. \tag{56}$$

Since Equation (51) is linear, by the superposition principle, we can combine the solutions in Equation (56) to obtain

$$w(r) = Ae^{-\sqrt{a}r} + Be^{\sqrt{a}r}. \tag{57}$$

Alternatively, in the special case where $a = 0$, Equation (51) can be integrated twice to obtain

$$w(r) = Ar + B, \tag{58}$$

where $A$ and $B$ are the constants of integration.

Finally, we can apply the inverse transformation to obtain the solutions

$$u(x,t) = e^{at} w, \tag{59}$$

$$= \begin{cases} Ae^{at-\sqrt{a}x} + Be^{at+\sqrt{a}x}, & a \neq 0, \\ (Ax + B), & a = 0. \end{cases} \tag{60}$$

## 1.2 Summarizing the method of symmetry reduction

Typically, the method of symmetry reduction of a differential equation can be separated into the following steps

1. Identify the group of transformations,
2. Construct the characteristic system,
3. Derive a new set of coordinates,
4. Transform the transformations to the new coordinate system,
5. Construct the new characteristic system,
6. Transform to the target equation,
7. Simplify to the reduced equation,
8. Solve for the solution and apply the inverse map.

## 1.3 Example 2: The heat equation - another symmetry

The heat equation given in Equation (3) also possesses the following translational symmetry

$$\tilde{x} = x + \alpha\epsilon, \qquad \tilde{t} = t + \epsilon, \qquad \tilde{u} = u, \tag{61}$$

where $\alpha$ is a constant.

### 1.3.1 Exercise A1

Show that the symmetry given by Equation (61) is also a symmetry of the heat equation.

### 1.3.2 Exercise A2

Construct the characteristic system for the symmetry given by Equation (61) and solve it to determine a new set of variables.

### 1.3.3 Exercise A3

Construct the corresponding transformations in $rsw$-space and from these write down the new characteristic system.

### 1.3.4 Exercise A4

Derive the target equation by expressing the heat equation in terms of the new variables.

### 1.3.5 Exercise A5

Simplify the target equation to the reduced equation using knowledge of the $rsw$-characteristic system.

### 1.3.6 Exercise A6

Solve the reduced equation and transform the solution back to the original variables in order to obtain a solution to the heat equation.

### 1.3.7 Answer A1

In order to show that Equation (61) represents a symmetry of the heat equation, we substitute for $u$ and then evaluate the resulting equation to determine if the symmetry criterion, Equation (2), is satisfied. Substituting yields

$$\frac{\partial \tilde{u}}{\partial t} - \frac{\partial^2 \tilde{u}}{\partial x^2} = 0. \tag{62}$$

To evaluate this equation we treat $\tilde{u}$ as a function of $\tilde{x}$ and $\tilde{t}$. The first-order derivatives are therefore

$$\frac{\partial \tilde{u}}{\partial t} = \frac{\partial \tilde{x}}{\partial t}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{t}}{\partial t}\frac{\partial \tilde{u}}{\partial \tilde{t}}, \tag{63}$$

$$\frac{\partial \tilde{u}}{\partial x} = \frac{\partial \tilde{x}}{\partial x}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{t}}{\partial x}\frac{\partial \tilde{u}}{\partial \tilde{t}}. \tag{64}$$

From the transformations given in Equation (61)

$$\frac{\partial \tilde{x}}{\partial x} = 1, \quad \frac{\partial \tilde{x}}{\partial t} = 0, \quad \frac{\partial \tilde{t}}{\partial x} = 0, \qquad \frac{\partial \tilde{t}}{\partial t} = 1, \tag{65}$$

therefore

$$\frac{\partial \tilde{u}}{\partial t} = \frac{\partial \tilde{u}}{\partial \tilde{t}}, \tag{66}$$

$$\frac{\partial \tilde{u}}{\partial x} = \frac{\partial \tilde{u}}{\partial \tilde{x}}. \tag{67}$$

Finally, we compute

$$\frac{\partial^2 \tilde{u}}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \tilde{u}}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial \tilde{u}}{\partial \tilde{x}}\right) = \frac{\partial \tilde{x}}{\partial x}\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} = \frac{\partial^2 \tilde{u}}{\tilde{x}^2}. \tag{68}$$

Putting these results together gives

$$\frac{\partial \tilde{u}}{\partial \tilde{t}} - \frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} = \Delta(\tilde{\mathbf{x}}, \tilde{\mathbf{u}}, \tilde{\mathbf{p}}) = 0. \tag{69}$$

Thus Equation (2) is satisfied and Equation (61) represents another symmetry of the heat equation.

#### 1.3.8 Answer A2

First expand the transformations as Taylor series around the identity transformation. This requires

$$\left.\frac{\partial \tilde{x}}{\partial \epsilon}\right|_{\epsilon=0} = \alpha, \qquad \left.\frac{\partial \tilde{t}}{\partial \epsilon}\right|_{\epsilon=0} = 1, \qquad \left.\frac{\partial \tilde{u}}{\partial \epsilon}\right|_{\epsilon=0} = 0, \tag{70}$$

therefore the Taylor series are

$$\tilde{x} = x + \epsilon\alpha + \mathcal{O}(\epsilon^2), \qquad \tilde{t} = t + \epsilon + \mathcal{O}(\epsilon^2), \qquad \tilde{u} = u + \mathcal{O}(\epsilon^2). \tag{71}$$

Subtracting the original variables from the transformed variables, dividing by $\epsilon$ and taking the limit as $\epsilon \to 0$ then yields

$$\frac{\mathrm{d}x}{\mathrm{d}\epsilon} = \alpha, \qquad \frac{\mathrm{d}t}{\mathrm{d}\epsilon} = 1, \qquad \frac{\mathrm{d}u}{\mathrm{d}\epsilon} = 0. \tag{72}$$

Solving each for $\mathrm{d}\epsilon$ then yields the characteristic system

$$\frac{\mathrm{d}x}{\alpha} = \frac{\mathrm{d}t}{1} = \frac{\mathrm{d}u}{0} = \mathrm{d}\epsilon. \tag{73}$$

Next we solve the characteristic system to determine the new variables. From the equation consisting of the first two members of the characteristic system we obtain

$$\int \mathrm{d}x = \int \alpha \mathrm{d}t, \tag{74}$$

therefore

$$x = \alpha t + r, \quad \text{thus} \quad r = x - \alpha t, \tag{75}$$

where $r$ is the constant of integration and therefore the first new coordinate. Similarly, from the first and third members,

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 0, \tag{76}$$

thus

$$w = u, \tag{77}$$

where $w$ is the constant of integration and the second new coordinate. Finally, integrating

$$\frac{\mathrm{d}x}{\alpha} = \mathrm{d}\epsilon, \tag{78}$$

yields

$$\frac{x}{\alpha} = \epsilon + c, \tag{79}$$

where $c$ is the constant of integration. Consequently we select the shift coordinate to be

$$s = \frac{x}{\alpha}. \tag{80}$$

In summary, the new coordinates are

$$r = x - \alpha t, \qquad s = \frac{x}{\alpha}, \qquad w = u. \tag{81}$$

#### 1.3.9 Answer A3

The transformations in $rsw$-space are determined as follows

$$\tilde{r} = \tilde{x} - \alpha\tilde{t} = x + \alpha\epsilon - \alpha(t+\epsilon) = x - \alpha t = r, \qquad \text{(Identity)}, \tag{82}$$

$$\tilde{s} = \frac{\tilde{x}}{\alpha} = \frac{x+\alpha\epsilon}{\alpha} = \frac{x}{\alpha} + \epsilon = s + \epsilon, \qquad \text{(Translation)}, \tag{83}$$

$$\tilde{w} = \tilde{u} = u = w, \qquad \text{(Identity)}. \tag{84}$$

The $rsw$ characteristic system is therefore

$$\frac{\mathrm{d}r}{0} = \frac{\mathrm{d}s}{1} = \frac{\mathrm{d}w}{0} = \mathrm{d}\epsilon. \tag{85}$$

### 1.3.10 Answer A4

The target equation is obtained by first substituting for $u$ to obtain

$$\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} = 0. \tag{86}$$

Treating $w$ as a function of $r$ and $s$, the first order derivatives are found to be

$$\frac{\partial w}{\partial t} = \frac{\partial r}{\partial t}\frac{\partial w}{\partial r} + \frac{\partial s}{\partial t}\frac{\partial w}{\partial s}, \tag{87}$$

$$\frac{\partial w}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial w}{\partial r} + \frac{\partial s}{\partial x}\frac{\partial w}{\partial s}. \tag{88}$$

From the transformations given in Equation (61) we determine

$$\frac{\partial r}{\partial t} = -\alpha, \qquad \frac{\partial r}{\partial x} = 1, \qquad \frac{\partial s}{\partial t} = 0, \qquad \frac{\partial s}{\partial x} = \frac{1}{\alpha}. \tag{89}$$

Therefore

$$\frac{\partial w}{\partial t} = -\alpha\frac{\partial w}{\partial r}, \tag{90}$$

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r} + \frac{1}{\alpha}\frac{\partial w}{\partial s}. \tag{91}$$

The second spatial derivative is then

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial r} + \frac{1}{\alpha}\frac{\partial w}{\partial s}\right), \tag{92}$$

$$= \frac{\partial r}{\partial x}\frac{\partial^2 w}{\partial r^2} + \frac{\partial s}{\partial x}\frac{\partial^2 w}{\partial r \partial s} + \frac{1}{\alpha}\left(\frac{\partial r}{\partial x}\frac{\partial^2 w}{\partial r \partial s} + \frac{\partial s}{\partial x}\frac{\partial^2 w}{\partial s^2}\right), \tag{93}$$

$$= \frac{\partial^2 w}{\partial r^2} + \frac{2}{\alpha}\frac{\partial^2 w}{\partial r \partial s} + \frac{1}{\alpha^2}\frac{\partial^2 w}{\partial s^2}. \tag{94}$$

Combining the results, the target equation is

$$-\alpha\frac{\partial w}{\partial r} - \frac{\partial^2 w}{\partial r^2} - \frac{2}{\alpha}\frac{\partial^2 w}{\partial r \partial s} - \frac{1}{\alpha^2}\frac{\partial^2 w}{\partial s^2} = 0. \tag{95}$$

### 1.3.11 Answer A5

The target equation is simplified using information from the $rsw$-characteristic system. If $w$ is a function of $r$ and $s$ then

$$\frac{\mathrm{d}w}{\mathrm{d}\epsilon} = \frac{\partial w}{\partial r}\frac{\mathrm{d}r}{\mathrm{d}\epsilon} + \frac{\partial w}{\partial s}\frac{\mathrm{d}s}{\mathrm{d}\epsilon}. \tag{96}$$

From the characteristic system constructed in Equation (85) we have

$$\frac{\mathrm{d}w}{\mathrm{d}\epsilon} = 0, \qquad \frac{\mathrm{d}r}{\mathrm{d}\epsilon} = 0, \qquad \frac{\mathrm{d}s}{\mathrm{d}\epsilon} = 1. \tag{97}$$

Substituting into Equation (96) yields

$$\frac{\partial w}{\partial s} = 0. \tag{98}$$

Consequently, we can reduce the target equation to

$$-\alpha\frac{\partial w}{\partial r} - \frac{\partial^2 w}{\partial r^2} = 0, \tag{99}$$

along the direction $\mathrm{d}\epsilon$.

### 1.3.12 Answer A6

The reduced equation, Equation (99), is a second-order linear ordinary differential equation. A solution can be determined by assuming

$$w(r) \propto e^{\lambda r}, \tag{100}$$

for some constant $\lambda$. Under this assumption

$$\frac{\partial w}{\partial r} = \lambda e^{\lambda r}, \qquad \frac{\partial^2 w}{\partial r^2} = \lambda^2 e^{\lambda r}. \tag{101}$$

Substituting into the reduced equation yields

$$-e^{\lambda r}\left(\alpha\lambda + \lambda^2\right) = 0, \tag{102}$$

which is satisfied for $\lambda = -\alpha$ or $\lambda = 0$. The two resulting solutions are therefore

$$w(r) = Ae^{-\alpha r}, \quad \text{or} \quad w(r) = B, \tag{103}$$

where $A$ and $B$ are constants. Since Equation (99) is linear, by the superposition principle, we can combine these two solutions into

$$w(r) = Ae^{-\alpha r} + B. \tag{104}$$

Finally, mapping back to the original coordinate system, the solution to the heat equation is

$$u(x,t) = Ae^{-\alpha(x-\alpha t)} + B. \tag{105}$$

## 1.4 Example 3: The two-dimensional wave equation

The two-dimensional wave equation is given by

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0. \tag{106}$$

In this example

$$\mathbf{x} = [t, x, y], \qquad \mathbf{u} = u, \qquad \mathbf{p} = \left[\frac{\partial^2 u}{\partial t^2}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}\right]. \tag{107}$$

Equation (106) admits the symmetry

$$\tilde{x} = x\cos\epsilon - y\sin\epsilon, \qquad \tilde{y} = x\sin\epsilon + y\cos\epsilon \qquad \tilde{t} = t, \qquad \tilde{u} = u. \tag{108}$$

**Remark 2.** *Again, note that the identity transformation corresponds to $\epsilon = 0$.*

**Remark 3.** *For additional symmetries admitted by the two-dimensional wave equation, see [6, Olver p.124].*

### 1.4.1 Exercise B1

Show that the group of transformations given in Equation (108) is a symmetry group of the two-dimensional wave equation.

### 1.4.2 Exercise B2

Construct the characteristic system for the symmetry given by Equation (108) and solve it to determine a new set of variables.

### 1.4.3 Exercise B3

Construct the corresponding transformations in $qrsw$-space and from these write down the characteristic system.

### 1.4.4 Exercise B4

Construct the target equation by expressing the two-dimensional wave equation in terms of the new variables.

### 1.4.5 Exercise B5

Simplify the target equation to the reduced equation using knowledge of the $qrsw$-characteristic system.

### 1.4.6 Answer B1

From the transformations given in Equation (108) we can derive the following first order derivatives

$$\frac{\partial \tilde{x}}{\partial x} = \cos\epsilon, \qquad \frac{\partial \tilde{x}}{\partial y} = -\sin\epsilon, \qquad \frac{\partial \tilde{x}}{\partial t} = 0, \tag{109}$$

$$\frac{\partial \tilde{y}}{\partial x} = \sin\epsilon, \qquad \frac{\partial \tilde{y}}{\partial y} = \cos\epsilon, \qquad \frac{\partial \tilde{y}}{\partial t} = 0, \tag{110}$$

$$\frac{\partial \tilde{t}}{\partial x} = 0, \qquad \frac{\partial \tilde{t}}{\partial y} = 0, \qquad \frac{\partial \tilde{t}}{\partial t} = 1. \tag{111}$$

$$\frac{\partial \tilde{u}}{\partial t} = \frac{\partial \tilde{t}}{\partial t}\frac{\partial \tilde{u}}{\partial \tilde{t}} + \frac{\partial \tilde{x}}{\partial t}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{y}}{\partial t}\frac{\partial \tilde{u}}{\partial \tilde{y}} = \frac{\partial \tilde{u}}{\partial \tilde{t}}, \tag{112}$$

$$\frac{\partial \tilde{u}}{\partial x} = \frac{\partial \tilde{t}}{\partial x}\frac{\partial \tilde{u}}{\partial \tilde{t}} + \frac{\partial \tilde{x}}{\partial x}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{y}}{\partial x}\frac{\partial \tilde{u}}{\partial \tilde{y}} = \cos\epsilon\frac{\partial \tilde{u}}{\partial \tilde{x}} + \sin\epsilon\frac{\partial \tilde{u}}{\partial \tilde{y}}, \tag{113}$$

$$\frac{\partial \tilde{u}}{\partial y} = \frac{\partial \tilde{t}}{\partial y}\frac{\partial \tilde{u}}{\partial \tilde{t}} + \frac{\partial \tilde{x}}{\partial y}\frac{\partial \tilde{u}}{\partial \tilde{y}} + \frac{\partial \tilde{y}}{\partial y}\frac{\partial \tilde{u}}{\partial \tilde{y}} = -\sin\epsilon\frac{\partial \tilde{u}}{\partial \tilde{x}} + \cos\epsilon\frac{\partial \tilde{u}}{\partial \tilde{y}}. \tag{114}$$

Next, we substitute the transformation $u = \tilde{u}$ into the two-dimensional wave equation to obtain

$$\frac{\partial^2 \tilde{u}}{\partial t^2} - \frac{\partial^2 \tilde{u}}{\partial x^2} - \frac{\partial^2 \tilde{u}}{\partial y^2} = 0. \tag{115}$$

Each of the terms are then expanded using the chain rule treating $\tilde{u}$ as a function of $\tilde{t}$, $\tilde{x}$ and $\tilde{y}$

$$\frac{\partial^2 \tilde{u}}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial \tilde{u}}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial \tilde{u}}{\partial \tilde{t}}\right) = \frac{\partial \tilde{t}}{\partial t}\frac{\partial^2 \tilde{u}}{\partial \tilde{t}^2} = \frac{\partial^2 \tilde{u}}{\partial \tilde{t}^2}, \tag{116}$$

$$\frac{\partial^2 \tilde{u}}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \tilde{u}}{\partial x}\right) = \frac{\partial}{\partial x}\left(\cos\epsilon\frac{\partial \tilde{u}}{\partial \tilde{x}} + \sin\epsilon\frac{\partial \tilde{u}}{\partial \tilde{y}}\right), \tag{117}$$

$$= \cos\epsilon\left(\frac{\partial \tilde{t}}{\partial x}\frac{\partial^2 \tilde{u}}{\partial \tilde{t}\partial \tilde{x}} + \frac{\partial \tilde{x}}{\partial x}\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} + \frac{\partial \tilde{y}}{\partial x}\frac{\partial^2 \tilde{u}}{\partial \tilde{y}\partial \tilde{x}}\right) + \sin\epsilon\left(\frac{\partial \tilde{t}}{\partial x}\frac{\partial^2 \tilde{u}}{\partial \tilde{t}\partial \tilde{y}} + \frac{\partial \tilde{x}}{\partial x}\frac{\partial^2 \tilde{u}}{\partial \tilde{x}\partial \tilde{y}} + \frac{\partial \tilde{y}}{\partial x}\frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}\right), \tag{118}$$

$$= \cos^2\epsilon\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} + 2\sin\epsilon\cos\epsilon\frac{\partial^2 \tilde{u}}{\partial \tilde{x}\partial \tilde{y}} + \sin^2\epsilon\frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}. \tag{119}$$

Similarly,

$$\frac{\partial^2 \tilde{u}}{\partial y^2} = \sin^2\epsilon\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} - 2\sin\epsilon\cos\epsilon\frac{\partial^2 \tilde{u}}{\partial \tilde{x}\partial \tilde{y}} + \cos^2\epsilon\frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}. \tag{120}$$

Bringing these results together we have

$$\frac{\partial^2 \tilde{u}}{\partial t^2} - \frac{\partial^2 \tilde{u}}{\partial x^2} - \frac{\partial^2 \tilde{u}}{\partial y^2} = \frac{\partial^2 \tilde{u}}{\partial \tilde{t}^2} - \left(\sin^2\epsilon + \cos^2\epsilon\right)\left(\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}\right) = 0, \tag{121}$$

Thus,

$$\frac{\partial^2 \tilde{u}}{\partial \tilde{t}^2} - \frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} - \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2} = \Delta(\tilde{\mathbf{x}}, \tilde{\mathbf{u}}, \tilde{\mathbf{p}}) = 0, \tag{122}$$

and Equation (2) is satisfied.

### 1.4.7 Answer B2

First, expand the transformations as Taylor series around the identity transformation, $\epsilon = 0$, which requires

$$\left.\frac{\partial \tilde{x}}{\partial \epsilon}\right|_{\epsilon=0} = \left.(-x\sin\epsilon - y\cos\epsilon)\right|_{\epsilon=0} = -y, \tag{123}$$

$$\left.\frac{\partial \tilde{y}}{\partial \epsilon}\right|_{\epsilon=0} = \left.(x\cos\epsilon - y\sin\epsilon)\right|_{\epsilon=0} = x, \tag{124}$$

$$\left.\frac{\partial \tilde{t}}{\partial \epsilon}\right|_{\epsilon=0} = 0, \tag{125}$$

$$\left.\frac{\partial \tilde{u}}{\partial \epsilon}\right|_{\epsilon=0} = 0. \tag{126}$$

The Taylor series are

$$\tilde{x} = x - \epsilon y + \mathcal{O}(\epsilon^2), \qquad \tilde{y} = y + \epsilon x + \mathcal{O}(\epsilon^2), \qquad \tilde{t} = t + \mathcal{O}(\epsilon^2), \qquad \tilde{u} = u + \mathcal{O}(\epsilon^2). \tag{127}$$

Subtracting the original variables from the transformed variables, diving by $\epsilon$ and taking the limit as $\epsilon \to 0$ yields the characteristic system

$$\frac{\mathrm{d}x}{-y} = \frac{\mathrm{d}y}{x} = \frac{\mathrm{d}t}{0} = \frac{\mathrm{d}u}{0} = \mathrm{d}\epsilon. \tag{128}$$

Solving the equation comprised of the first two members

$$\int x\mathrm{d}x = -\int y\mathrm{d}y, \tag{129}$$

yields

$$q = x^2 + y^2, \tag{130}$$

where $q$ is the constant of integration and the first new variable. Solving the equation from the first and third members

$$\frac{\mathrm{d}t}{\mathrm{d}x} = 0, \tag{131}$$

yields

$$r = t, \tag{132}$$

where $r$ is the constant of integration and the second new variable. Similarly, from the first and fourth members

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 0, \tag{133}$$

thus

$$w = u, \tag{134}$$

where $w$ is the integration constant and the third new variable. Finally, we solve

$$\frac{\mathrm{d}x}{-y} = \mathrm{d}\epsilon, \tag{135}$$

by eliminating $y$ using $y = -(q - x^2)^{\frac{1}{2}}$. This yields

$$\frac{\mathrm{d}}{\sqrt{q - x^2}} = \mathrm{d}\epsilon, \tag{136}$$

which upon integration gives

$$-\frac{(q - x^2)^{\frac{1}{2}}}{x} = \frac{y}{x} = \epsilon + c. \tag{137}$$

At this point, we recognise

$$\frac{y}{x} = \tan\theta, \tag{138}$$

where $\theta$ is the angle corresponding to the point $(x, y)$ on the circle with radius $\sqrt{q}$ centered at the origin. Therefore, we select the shift coordinate for this example to be

$$s = \theta = \arctan\frac{y}{x}, \tag{139}$$

since it is a more natural coordinate choice for this example. Summarizing, the new variables are

$$q = x^2 + y^2, \qquad r = t, \qquad s = \theta = \arctan\frac{y}{x}, \qquad w = u. \tag{140}$$

### 1.4.8 Answer B3

Next, lets check how the transformations transfer to $qrsw$-space. First

$$\begin{aligned}
\tilde{q} &= \tilde{x}^2 + \tilde{y}^2, && (141)\\
&= (x\cos\epsilon - y\sin\epsilon)^2 + (x\sin\epsilon + y\cos\epsilon)^2,\\
&= x^2\cos^2\epsilon - 2xy\cos\epsilon\sin\epsilon + y^2\sin^2\epsilon + x^2\sin^2\epsilon + 2xy\cos\epsilon\sin\epsilon + y^2\cos^2\epsilon,\\
&= (x^2+y^2)(\sin^2\epsilon + \cos^2\epsilon),\\
&= x^2 + y^2 = q, \qquad \text{(Identity)}, && (142)\\
\tilde{r} &= \tilde{t} = t = r, \qquad \text{(Identity)}, && (143)\\
\tilde{w} &= \tilde{u} = u = w, \qquad \text{(Identity)}, && (144)\\
\tan\tilde{\theta} &= \frac{\tilde{y}}{\tilde{x}} = \frac{x\sin\epsilon + y\cos\epsilon}{x\cos\epsilon - y\sin\epsilon}, && (145)\\
&= \frac{\sqrt{q}\cos\theta\sin\epsilon + \sqrt{q}\sin\theta\cos\epsilon}{\sqrt{q}\cos\theta\cos\epsilon - \sqrt{q}\sin\theta\sin\epsilon}, && (146)\\
&= \frac{\sin(\theta+\epsilon)}{\cos(\theta+\epsilon)}, && (147)\\
&= \tan(\theta+\epsilon), \qquad \text{(Translation)}. && (148)
\end{aligned}$$

**Remark 4.** *We applied the following trigonometric identities to derive the final transformation*

$$\begin{aligned}
x &= \sqrt{q}\cos\theta, && (149)\\
y &= \sqrt{q}\sin\theta, && (150)\\
\sin(\alpha+\beta) &= \cos\alpha\sin\beta + \sin\alpha\cos\beta, && (151)\\
\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta, && (152)\\
\tan\alpha &= \frac{\sin\alpha}{\cos\alpha}. && (153)
\end{aligned}$$

Thus the characteristic system in $qrsw$-space is

$$\frac{\mathrm{d}q}{0} = \frac{\mathrm{d}r}{0} = \frac{\mathrm{d}s}{1} = \frac{\mathrm{d}w}{0} = \mathrm{d}\epsilon. \tag{154}$$

### 1.4.9 Answer B4

First, we determine the set of first order derivatives we will require for the substitution

$$\begin{aligned}
&\frac{\partial q}{\partial t} = 0, && \frac{\partial q}{\partial x} = 2x, && \frac{\partial q}{\partial y} = 2y, && (155)\\
&\frac{\partial r}{\partial t} = 1, && \frac{\partial r}{\partial x} = 0, && \frac{\partial r}{\partial y} = 0, && (156)\\
&\frac{\partial s}{\partial t} = 0, && \frac{\partial s}{\partial x} = -\frac{y}{q}, && \frac{\partial s}{\partial y} = \frac{x}{q}, && (157)
\end{aligned}$$

$$\frac{\partial w}{\partial t} = \frac{\partial q}{\partial t}\frac{\partial w}{\partial q} + \frac{\partial r}{\partial t}\frac{\partial w}{\partial r} + \frac{\partial s}{\partial t}\frac{\partial w}{\partial s}, \tag{158}$$

$$= \frac{\partial w}{\partial r}, \tag{159}$$

$$\frac{\partial w}{\partial x} = \frac{\partial q}{\partial x}\frac{\partial w}{\partial q} + \frac{\partial r}{\partial x}\frac{\partial w}{\partial r} + \frac{\partial s}{\partial x}\frac{\partial w}{\partial s}, \tag{160}$$

$$= 2x\frac{\partial w}{\partial q} - \frac{y}{q}\frac{\partial w}{\partial s}, \tag{161}$$

$$\frac{\partial w}{\partial y} = \frac{\partial q}{\partial y}\frac{\partial w}{\partial q} + \frac{\partial r}{\partial y}\frac{\partial w}{\partial r} + \frac{\partial s}{\partial y}\frac{\partial w}{\partial s}, \tag{162}$$

$$= 2y\frac{\partial w}{\partial q} + \frac{x}{q}\frac{\partial w}{\partial s}. \tag{163}$$

where we have treated $w$ as a function of $q$, $r$ and $s$.

We can now evaluate the second-order derivatives

$$\frac{\partial^2 w}{\partial^2 t} = \frac{\partial}{\partial t}\left(\frac{\partial w}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial w}{\partial r}\right) = \frac{\partial^2 w}{\partial r^2}, \tag{164}$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right) = \frac{\partial}{\partial x}\left(2x\frac{\partial w}{\partial q} - \frac{y}{q}\frac{\partial w}{\partial s}\right), \tag{165}$$

$$= 2\frac{\partial w}{\partial q} + 2x\left(\frac{\partial q}{\partial x}\frac{\partial^2 w}{\partial q^2} + \frac{\partial r}{\partial x}\frac{\partial^2 w}{\partial q \partial r} + \frac{\partial s}{\partial x}\frac{\partial^2 w}{\partial q \partial s}\right) \tag{166}$$

$$+ \frac{y}{q^2}\frac{\partial q}{\partial x}\frac{\partial w}{\partial s} - \frac{y}{q}\left(\frac{\partial q}{\partial x}\frac{\partial^2 w}{\partial q \partial s} + \frac{\partial r}{\partial x}\frac{\partial^2 w}{\partial r \partial s} + \frac{\partial s}{\partial x}\frac{\partial^2 w}{\partial s^2}\right), \tag{167}$$

$$= 2\frac{\partial w}{\partial q} + 4x^2\frac{\partial^2 w}{\partial q^2} - \frac{4xy}{q}\frac{\partial^2 w}{\partial q \partial s} + \frac{2xy}{q^2}\frac{\partial w}{\partial s} + \frac{y^2}{q^2}\frac{\partial^2 w}{\partial s^2}. \tag{168}$$

Similarly,

$$\frac{\partial^2 w}{\partial y^2} = 2\frac{\partial w}{\partial q} + 4y^2\frac{\partial^2 w}{\partial q^2} + \frac{4xy}{q}\frac{\partial^2 w}{\partial q \partial s} - \frac{2xy}{q^2}\frac{\partial w}{\partial s} + \frac{x^2}{q^2}\frac{\partial^2 w}{\partial s^2}. \tag{169}$$

Combining these results, the target equation is

$$\frac{\partial^2 w}{\partial r^2} - 4\left(\frac{\partial w}{\partial q} + q\frac{\partial^2 w}{\partial q^2}\right) - \frac{1}{q}\frac{\partial^2 w}{\partial s^2} = 0. \tag{170}$$

### 1.4.10 Answer B5

In order to reduce the target equation we refer back to the characteristic system derived in Equation (154). The total derivative of $w$ with respect to $\epsilon$ is given by

$$\frac{\mathrm{d}w}{\mathrm{d}\epsilon} = \frac{\partial w}{\partial q}\frac{\mathrm{d}q}{\mathrm{d}\epsilon} + \frac{\partial w}{\partial r}\frac{\mathrm{d}r}{\mathrm{d}\epsilon} + \frac{\partial w}{\partial s}\frac{\mathrm{d}s}{\mathrm{d}\epsilon}. \tag{171}$$

From the characteristic equation we have

$$\frac{\mathrm{d}w}{\mathrm{d}\epsilon} = 0, \quad \frac{\mathrm{d}q}{\mathrm{d}\epsilon} = 0, \quad \frac{\mathrm{d}r}{\mathrm{d}\epsilon} = 0, \quad \frac{\mathrm{d}s}{\mathrm{d}\epsilon} = 1. \tag{172}$$

Thus, we infer

$$\frac{\partial w}{\partial s} = 0, \tag{173}$$

and the target equation can be reduced to

$$\frac{\partial^2 w}{\partial r^2} - 4\left(\frac{\partial w}{\partial q} + q\frac{\partial^2 w}{\partial q^2}\right) = 0, \tag{174}$$

along the direction $\mathrm{d}\epsilon$.

## 2 Generating new solutions using symmetries

In section 1.1, we demonstrated that the heat equation possessed the symmetry given by Equation (5) and used it to determine the solution prescribed in Equation (59). It can also be shown that the heat equation admits a Galilean boost type symmetry of the form

$$\tilde{x} = x + 2\sqrt{a}t, \qquad \tilde{t} = t, \qquad \tilde{u} = ue^{-(at+\sqrt{a}x)}. \tag{175}$$

We can use this symmetry to determine a new solution to the heat equation. This is achieved simply by substituting the transformations given by Equation (175) into the old solution. For the case when $a > 0$, the solution is

$$u(x,t) = Ae^{at-\sqrt{a}x} + Be^{at+\sqrt{a}x}. \tag{176}$$

The new transformed solution is then given by

$$\tilde{u} = ue^{-(at+\sqrt{a}x)}, \tag{177}$$

$$= \left(Ae^{at-\sqrt{a}x} + Be^{at+\sqrt{a}x}\right)e^{-(at+\sqrt{a}x)}, \tag{178}$$

$$= Ae^{-2\sqrt{a}x} + B. \tag{179}$$

We need to express this solution in terms of the tilded variables. Thus, substituting for $x$ and $t$ using

$$x = \tilde{x} - 2\sqrt{a}t, \qquad t = \tilde{t}, \tag{180}$$

yields

$$\tilde{u}(\tilde{x},\tilde{t}) = Ae^{-2\sqrt{a}(\tilde{x}-2\sqrt{a}\tilde{t})} + B. \tag{181}$$

Therefore, the new solution to the heat equation is given by

$$u(x,t) = Ae^{-2\sqrt{a}(x-2\sqrt{a}t)} + B. \tag{182}$$

This solution is of the form

$$u(x,t) = f(x - \alpha t), \tag{183}$$

where $\alpha = 2\sqrt{a}$. This is a travelling wave solution analogous to Equation (105) determined in Section 1.3 using the translation symmetry.

## 3 A set of definitions

**Definition 1.** *A* group, *$\mathcal{G}$, is a set of elements with a law of composition, $\phi$, between elements satisfying the following axioms:*

*(i)* Closure property. *For any elements, $a$ and $b$ of $\mathcal{G}$, $\phi(a,b)$ is an element of $\mathcal{G}$.*

*(ii)* Associative property. *For any elements, $a$, $b$ and $c$ of $\mathcal{G}$:*

$$\phi(a,\phi(b,c)) = \phi(\phi(a,b),c). \tag{184}$$

*(iii)* Identity element. *There exists a unique element $e$ of $\mathcal{G}$ such that for any element $a$ of $\mathcal{G}$:*

$$\phi(a,e) = \phi(e,a) = a. \tag{185}$$

*(iv)* Inverse element. *For any element $a$ of $\mathcal{G}$, there exists a unique inverse element $a^{-1}$ in $\mathcal{G}$ such that*

$$\phi(a,a^{-1}) = \phi(a^{-1},a) = e. \tag{186}$$

*See [5, Bluman and Anco p.34].*

## 3.1 Examples of groups

- $\mathcal{G}$ is the set of all integers with the law of composition being addition $\phi(a,b) = a+b$. The identity element in this example is $e = 0$ and the inverse is $a^{-1} = -a$.
- $\mathcal{G}$ is the set of all positive real numbers with the law of composition being multiplication, $\phi(a,b) = a \cdot b$. The identity element is $e = 1$ and the inverse is $a^{-1} = 1/a$.

**Definition 2.** *A* one-parameter group of transformations*. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ lie in the region $D \subset \mathbf{R}^n$. The set of transformations*

$$\tilde{\mathbf{x}} = \mathbf{X}(\mathbf{x}; \epsilon), \tag{187}$$

*defined for each $\mathbf{x}$ in $D$ and parameter $\epsilon$ in set $S \subset \mathbf{R}$, with $\phi(\epsilon, \delta)$ defining a law of composition of the parameters $\epsilon$ and $\delta$ in $S$, forms a* one-parameter group of transformations *on $D$ if the following hold:*

*(i) For each $\epsilon$ in $S$, the transformations are one-to-one onto $D$.*

*(ii) $S$, with the law of composition, $\phi$, forms a group, $\mathcal{G}$.*

*(iii) For each $\mathbf{x}$ in $D$, $\tilde{\mathbf{x}} = \mathbf{x}$ when $\epsilon$ corresponds to the identity $e$, i.e.,*

$$\mathbf{X}(\mathbf{x}; e) = \mathbf{x}. \tag{188}$$

*(iv) If $\tilde{\mathbf{x}} = \mathbf{X}(\mathbf{x}; \epsilon)$ and $\tilde{\tilde{\mathbf{x}}} = \mathbf{X}(\tilde{\mathbf{x}}; \delta)$, then*

$$\tilde{\tilde{\mathbf{x}}} = \mathbf{X}(\mathbf{x}; \phi(\epsilon, \delta)). \tag{189}$$

*See [5, Bluman and Anco p.36].*

**Definition 3.** *A* one-parameter Lie group of transformations*. A one-parameter group of transformations defines a* one-parameter Lie group of transformations *if, in addition to satisfying axioms $(i)-(iv)$ of Definition 2, the following hold:*

*(v) $\epsilon$ is a continuous parameter, i.e., $S$ is an interval in $\mathbf{R}$. Without loss of generality, $\epsilon = 0$, corresponds to the identity element, $e$.*

*(vi) $\mathbf{X}$ is infinitely differentiable with respect to $\mathbf{x}$ in $D$ and an analytic function of $\epsilon$ in $S$.*

*(vii) $\phi(\epsilon, \delta)$ is an analytic function of $\epsilon$ and $\delta$, $\epsilon \in S$, $\delta \in S$.*

*See [5, Bluman and Anco p.36].*

## 3.2 Examples of one-parameter Lie groups of transformations

1. *A group of translations in the plane.*

$$\tilde{x} = x + \epsilon, \qquad \tilde{y} = y, \quad \epsilon \in \mathbf{R}. \tag{190}$$

For this example the law of composition is given by

$$\phi(\epsilon, \delta) = \epsilon + \delta, \tag{191}$$

the identity element is $\epsilon = e = 0$ and the inverse is $\epsilon^{-1} = -\epsilon$.

2. *A group of scalings in the plane.*

$$\tilde{x} = (1+\epsilon)x, \qquad \tilde{y} = (1+\epsilon)^2 y, \qquad -1 < \epsilon < \infty. \tag{192}$$

Here $\phi(\epsilon, \delta) = \epsilon + \delta + \epsilon \cdot \delta$ and the identity element is $\epsilon = e = 0$.

**Definition 4.** *A* symmetry *of a system of differential equations, $\Delta$, is a Lie group of transformations, $\mathcal{G}$, acting on the space of independent and dependent variables with the property that if $\mathbf{u} = \mathbf{f}(\mathbf{x})$ is a solution to $\Delta$, then $\tilde{\mathbf{u}} = \mathbf{X}(\mathbf{f}(\mathbf{x}); a)$ is also a solution of the system (whenever $\mathbf{X}(\mathbf{f}(\mathbf{x}); a)$) is defined.*
*See [6, Olver p.93]*

## 4 Follow up

The next place to look in order to learn more about the applications of symmetries to differential equations is [2, Albright et. al]. This primer introduces the concepts of vector fields, the group generator and the criterion of infinitesimal invariance. Each essential for the determination of symmetries of differential equations. Further examples of symmetry reductions of the inviscid Burgers's equation are also provided in [4] and examples for the heat equation, one-dimensional wave equation and Laplace's equation are given in [3]. Following this, the interested reader should consult [5] for a more comprehensive look at the subject or alternatively [6] for a very rigorous layout of the key aspects. Lastly, a brief introduction to the computation of symmetries using computational software is given in [1, Albright and McHardy] which also provides worked examples.

## References


[1] E. J. Albright and J. D. Mchardy, *Symmetry analysis for differential equations using symbolic computation*, LA-UR 18-26586, 2018.

[2] E. J. Albright, J. D. McHardy, S. D. Ramsey, and J. H. Schmidt, *Symmetry analysis of differential equations: A primer*, LA 13497, Los Alamos National Laboratory, 2017.

[3] J. Albright, *A note on deriving fundamental solutions to some canonical second-order equations using symmetry reduction*, LA-UR 18-26587, Los Alamos National Laboratory, 2018.

[4] J. Albright and J. D. McHardy, *The method of invariant coordinates for inviscid burgers's equation*, LA-UR (In preparation), Los Alamos National Laboratory, 2019.

[5] G. W. Bluman and S. C. Anco, *Symmetry and integration methods for differential equations*, vol. 154 of Applied Mathematical Sciences, Springer-Verlag, New York, 2002.

[6] P. J. Olver, *Applications of Lie groups to differential equations*, vol. 107 of Graduate Texts in Mathematics, Springer-Verlag, New York, second ed., 1993.